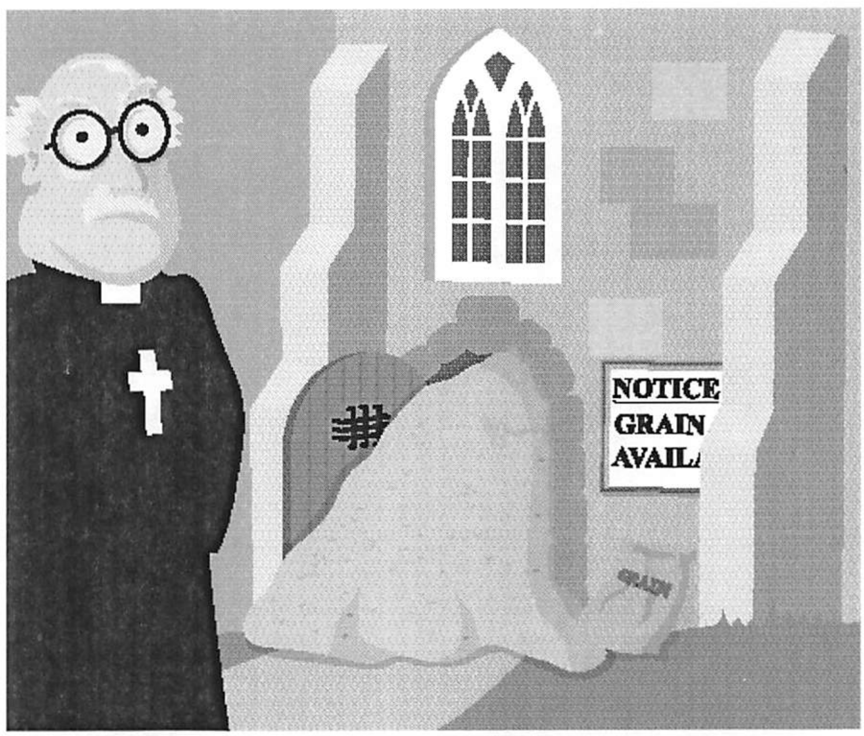
Good and Bad Properties of the Kelly Criterion

If your outlook is well extended, the Kelly criterion is the approach best suited to generating a fortune

This issue I would like to discuss good and bad properties of the Kelly expected log capital growth criterion and in the process lead into the next columns on hedge funds by discussing two of the great traders who ran unofficial hedge funds. The main advantages are that if your horizon is long enough then the Kelly criterion is the road, however bumpy, to the most wealth at the end and the fastest path to a given rather large fortune.

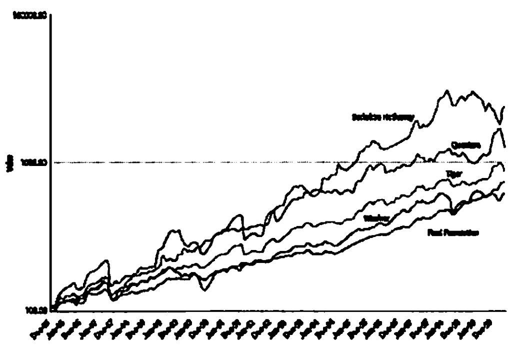
Thorp (1997) has shown that the great investor Warren Buffett's Berkshire Hathaway actually has had a growth path quite similar to full Kelly betting. Figure 1 shows this performance from 1985 to 2000 in comparison with other great funds. Buffett also had a great record from 1977 to 1985 turning 100 into 1429.87, and 65,852.40 in April 2000.



Mr Keynes believed God to be a large chicken, the Reverend surmised

Keynes was another Kelly-type bettor. His record running King's College Cambridge's Chest Fund is shown in Figure 2 versus the British market index for 1927 to 1945, data from Chua and Woodward (1983). Notice how much Keynes lost the first few years; obviously his academic brilliance and the recognition that he was facing a rather tough market kept him in this job. In total his geometric mean return beat the index by 10.01 per cent. Keynes was an aggressive investor with a beta of 1.78 versus the benchmark United Kingdom market return, a Sharpe ratio of 0.385, geometric mean returns of 9.12 per cent per year versus −0.89 per cent for the benchmark. Keynes had a yearly standard deviation of 29.28 per cent versus 12.55 per cent for

FIGURE 1: GROWTH OF ASSETS, LOG SCALE, VARIOUS HIGH PERFORMING FUNDS, 1985-2000. SOURCE: ZIEMBA (2003)



Source: *MacLean and Ziemba* (1999)

the benchmark. These returns do not include Keynes' (or the benchmark's) dividends and interest, which he used to pay the college expenses. These were 3 per cent per year. Kelly cowboys have their great returns and losses and embarrassments. Not covering a grain contract in time led to Keynes taking delivery and filling up the famous chapel. Fortunately it was big enough to fit in the grain and store it safely until it could be sold.

Keynes emphasized three principles of successful investments in his 1933 report:

1. A careful selection of a few investments (or a few types of investment) having regard to their cheapness in relation to their probable actual and potential intrinsic value over a period of years ahead and in relation to alternative investments at the time;

2. A steadfast holding of these in fairly large units through thick and thin, perhaps for several years until either they have fulfilled their promise or it is evident that they were purchased on a mistake; and

3. A balanced investment position, i.e., a variety of risks in spite of individual holdings being large, and if possible, opposed risks.

He really was a lot like Buffett with an emphasis on value, large holdings and patience.

In November 1919 Keynes was appointed second bursar. Up to this time King's College investments were only in fixed income trustee securities plus their own land and buildings. By June 1920 Keynes convinced the college to start a separate fund containing stocks, currency and commodity futures. Keynes became first bursar in 1924 and held this post which had final authority on investment decisions until his death in 1945.

And Keynes did not believe in market timing as he said:

"We have not proved able to take much advantage of a general systematic movement out of and into ordinary shares as a whole at different phases of the trade cycle. As a result of these experiences I am clear that the idea of wholesale shifts is for various reasons impracticable and indeed undesirable. Most of those who attempt to, sell too late and buy too late, and do both too often, incurring heavy expenses and developing too unsettled and speculative a state of mind, which, if it is widespread, has besides the grave social disadvantage of aggravating the scale of the fluctuations."

The main disadvantages result because the Kelly strategy is very very aggressive with huge bets that are larger and larger as the situ-

FIGURE 2: GRAPH OF THE PERFORMANCE OF THE CHEST FUND, 1927-1945

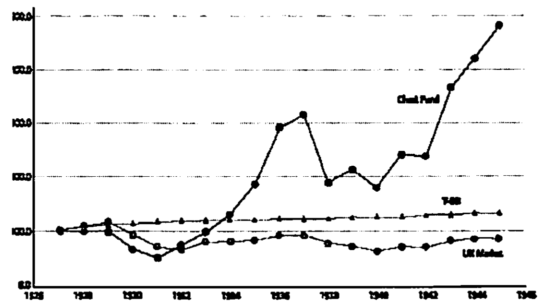
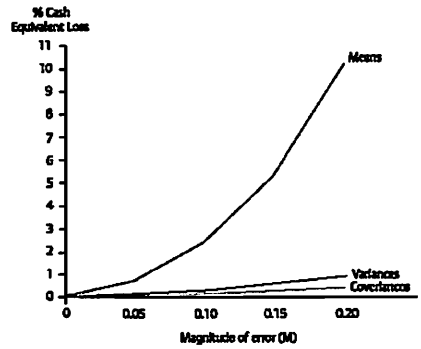


TABLE 1: AVERAGE RATIO OF CERTAINTY EQUIVALENT LOSS FOR ERRORS IN MEANS, VARIANCES AND COVARIANCES. SOURCE: CHOPRA AND ZIEMBA (1993)

t Risk Tolerance	Errors in Means vs Covariances	Errors in Means vs Variances	Errors in Variances vs Covariances
25	5.38	3.22	1.67
50	22.50	10.98	2.05
75	56.84	21.42	2.68
	↓	↓	↓
	20	10	2
	Error Mean	Error Var	Error Covar
	20	2	1

FIGURE 3: MEAN PERCENTAGE CASH EQUIVALENT LOSS DUE TO ERRORS IN INPUTS



ations are most attractive: recall that the bet is mean return divided by the odds of winning. As I repeatedly argue it's the mean that counts by far the most. There is about a 20–2:1 ratio of expected utility loss from similarly sized errors of means, variances and covariances, respectively. See Table 1 and Figure 3, Kallberg and Ziemba (1984) and Chopra and Ziemba (1993) for details. Returning to Buffett who gets the mean right, better than almost all, notice that the other funds he outperformed are not shabby ones at all. Indeed they are George Soros' Quantum, John Neff's Windsor, Julian Robertson's Tiger and the Ford Foundation, all of whom had great records as measured by the Sharpe ratio. Buffett made 32.07 per cent per year net from July 1977 to March 2000 versus 16.71 per cent for the S&P500. Wow! Those of us who like wealth prefer Warren's path but his higher standard deviation path (mostly winnings) leads to a lower Sharpe (normal distribution based) measure; see Siegel, Kjroner and Clifford (2001).

Kelly has essentially zero risk aversion since its Arrow-Pratt risk aversion index is $u''(w)/u'(w) = 1/w$, which is essentially zero. Hence it never pays to bet more than the Kelly strategy because then risk increases (lower security) and growth decreases so is stochastically dominated. As you bet more and more above the Kelly bet, its properties become worse and worse. When you bet exactly twice the Kelly bet, then the growth rate is zero plus the risk free rate.

If you bet more than double the Kelly criterion, then you will have a negative growth rate. With derivative positions one's bet changes continuously so a set of positions amounting to a

TABLE 2: KELLY CRITERION PROPERTIES

Good Maximizing $E[\log X]$ asymptotically maximizes the rate of asset growth See Breiman (1961), Algoet and Cover (1988)

Good The expected time to reach a preassigned goal is asymptotically as X increases least with a strategy maximizing $E[\log X_N]$. See Breiman (1961), Algoet and Cover (1988), Browne (1997).

Good Maximizing median $\log X$ See Ethier (1987)

Bad False property: If maximizing $E[\log X_N]$ almost certainly leads to a better outcome then the expected utility of its outcome exceeds that of any other rule provided N is sufficiently large. Counter Example: $u(x) = x$, $1/2 < p < 1$, Bernoulli trials $f = 1$ maximizes $E[U(x)]$ but $f = 2p - 1 < 1$ maximizes $E[\log X_N]$. See Samuelson (1971), Thorp (1975)

Good The $E[\log X]$ bettor never risks ruin. See Hakansson and Miller (1975)

Bad If the $E[\log X_N]$ bettor wins then loses or loses then wins, he is behind. The order of win and loss is immaterial for one, two, ..., sets of trials.
$(1+\gamma)(1-\gamma)X_0 = (1-\gamma^2)X_0 \leq X_0$.

Good The absolute amount bet is monotone in wealth. $(\delta E[\log X])/\delta W_0 > 0$.

Bad The bets are extremely large when the wager is favorable and the risk is very low. For single investment worlds, the optimal wager is proportional to the edge divided by the odds. Hence for low risk situations and corresponding low odds, the wager can be extremely large. For one such example, see Ziemba and Hausch (1986; 159-160). There, in the inaugural 1984 Breeders' Cup Classic $3 million race, the optimal fractional wager on the 3-5 shot Slew of Gold was 64%. (See also the 74% future bet on the January effect in a previous column.) Thorp and I actually made this place and show bet and won with a low fractional Kelly wager. Slew actually finished third but the second place horse Gate Dancer was disqualified and placed third. Luck (a good scenario) is also nice to have in betting markets. Wild Again won this race; the first great victory by the masterful jockey Pat Day.

Bad One overinvests when the problem data is uncertain. Investing more than the optimal capital growth wager is dominated in a growth-security sense. Hence, if the problem data provides probabilities, edges and odds that may be in error, then the suggested wager will be too large.

Bad The total amount wagered swamps the winnings – that is, there is much churning. Ethier and Tavar (1983) and Griffin (1985) show that the Expected Gain/E Bet is arbitrarily small and converges to zero in a Bernoulli game where one wins the expected fraction p of games.

Bad The unweighted average rate of return converges to half the arithmetic rate of return Related to property 5 this indicates that you do not seem to win as much as you expect. See Ethier and Tavar (1983) and Griffin (1985).

Bad Betting double the optimal Kelly bet reduces the growth rate of wealth to zero plus the risk free rate. See Stutzer (1998) and Janecek (1999) and the text here for proofs.

Good The $E[\log X]$ bettor is never behind any other bettor on average in 1, 2, ... trials. See Finkelstein and Whitley (1981)

Good The $E[\log X]$ bettor has an optimal myopic policy. He does not have to consider prior to subsequent investment opportunities. This is a crucially important result for practical use. Hakansson (1972) proved that the myopic policy obtains for dependent investments with the log utility function. For independent investments and power utility a myopic policy is optimal, see Mossin (1968).

Good The chance that an $E[\log X]$ wagerer will be ahead of any other wagerer after the first play is at least 50%. See Bell and Cover (1980).

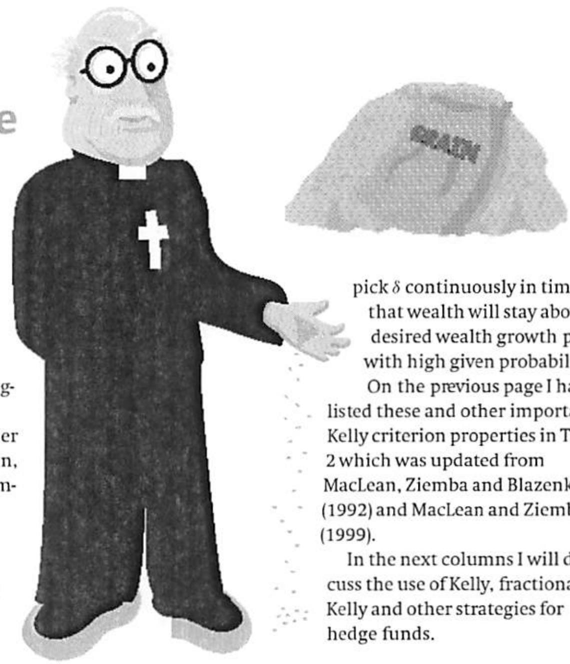
Good Simulation studies show that the ElogX bettor's fortune pulls way ahead of other strategies wealth for reasonable-sized samples. The key again is risk. See Ziemba and Hausch (1986). General formulas in Aucamp (1993).

Good If you wish to have higher security by trading it off for lower growth, then use a negative power utility function or fractional Kelly strategy. See MacLean, Ziemba and Li (2002). MacLean, Sanegre, Zhao and Ziemba (2002) show how to compute the coefficent to stay above a growth path with given probability.

Bad Despite its superior long-run growth properties, it is possible to have very poor return outcome. For example, making 700 wagers all of which have a 14% advantage, the least of which had a 19% chance of winning can turn $1000 into $18. But $1000 turns into $100,000 plus 16.6% of the time, see Ziemba and Hausch (1996).

Bad It can take a long time for a Kelly bettor to dominate an essentially different strategy. In fact this time may be without limit. Suppose $\mu_A = 20\%$, $\mu_B = 10\%$, $\sigma_A = \sigma_B = 10\%$. Then in five years A is ahead of B with 95% confidence. But if $\sigma_A = 20\%$, $\sigma_B = 10\%$ with the same means, it takes 157 years for A to beat B with 95% confidence. In coin tossing suppose game A has an edge of 1.0% and game B 1.1%. It takes two million trials to have an 84% chance that game A dominates game B, see Thorp (1997).

Not covering a grain contract in time led to Keynes taking delivery and filling up the famous chapel



small bet can turn into a large bet very quickly with market moves. Long Term Capital is a prime example of this overbetting leading to disaster but the phenomenon occurs all the time all over the world. Overbetting plus a bad scenario leads invariably to disaster.

Thus you must either bet Kelly or less. We call less than Kelly 'fractional Kelly', which is simply a blend of Kelly and cash. Consider the negative power utility function $\delta\omega^{\delta}$ for $\delta<0$. This utility function is concave and when $\delta \to 0$ it converges to log utility. As δ gets larger negatively, the investor is less aggressive since his Arrow-Pratt risk aversion is also higher. For a given δ an $\alpha = 1/(1-\delta)$ between 0 and 1, will provide the same portfolio when α is invested in the Kelly portfolio and $1-\alpha$ is invested in cash.

This result is correct for lognormal investments and approximately correct for other distributed assets; see MacLean, Ziemba and Li (2002). For example, half Kelly is $\delta = -1$ and quarter Kelly is $\delta = -3$. So if you want a less aggressive path, than Kelly then pick an appropriate δ. In a later column I will discuss a way to pick δ continuously in time so that wealth will stay above a desired wealth growth path with high given probability.

On the previous page I have listed these and other important Kelly criterion properties in Table 2 which was updated from MacLean, Ziemba and Blazenko (1992) and MacLean and Ziemba (1999).

In the next columns I will discuss the use of Kelly, fractional Kelly and other strategies for hedge funds.

APPENDIX

Selected References[1]

Chopra, V. and W. T. Ziemba (1993). The effect of errors in mean, variance and covariance estimates on optimal portfolio choice. *Journal of Portfolio Management*, 6-11.

Chua, J. H. and R. S. Woodward (1983) J. M. Keynes's investment performance; a note, *Journal of Finance* 38 (1): 232-235.

Kallberg, J. G. and W. T. Ziemba (1984). Mis-specifications in portfolio selection problems. In G. Bamberg and A. Spremann (Eds.), *Risk and Capital*, pp. 74-87. Springer Verlag, New York.

MacLean, L. and W. T. Ziemba (1999). Growth versus security tradeffs in dynamic investment analysis. *Annals of Operations Research* 85, 193-227.

MacLean, L.C., Ziemba, W.T., and Blazenko, G. (1992). Growth versus security in dynamic investment analysis. *Management Science*, 38, 1562-1585.

MacLean, Ziemba and Li (2002) Time to wealth goals in capital accumulation and the optimal trade off of growth versus security.

Siegel, L. B., K. F. Kjroner and S. W. Clifford (2001) The greatest return stories ever told. *The Journal of Investing*.

Thorp, E.O. (1997) The Kelly criterion in blackjack, sports betting, and the stock market, Presented at the 10th International Conference on Gambling and Risk Taking, Montreal, June.

Ziemba, W. T. (2003) *The Stochastic Programming Approach to Asset, Liability and Wealth Management*. AIMR (in press).

In the table

Algoet, P.H., and T.M. Cover (1988). Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *Annals of Probability*, 16.2, 876-898.

Aucamp, D. (1993) On the extensive number of plays to achieve superior performance with the geometric mean strategy. *Management Science* 39: 1163-1172.

Bell, R.M. and T.M. Cover (1980) Competitive optimality of logarithmic investment. *Math of Operations Research*.5: 161-166.

Breiman, L. (1961). Optimal gambling system for favorable games, in Proceedings 4th Berkeley Symposium on Mathematics, *Statistics and Probability* 1: 63-68.

Browne (1997) Survival and growth with a fixed liability: optimal portfolios in continuous time. *Math of Operations Research* 22: 468-493.

Ethier, S.N. (1987) The proportional bettor's fortune. Proceedings 7th International Conference on Gambling and Risk Taking. Department of Economics, University of Nevada, Reno.

Ethier, S.N. and S. Tavar (1983) The proportional bettor's return on investment. *Journal of Applied Probability* 20:. 563-573.

Finkelstein, M. and R. Whitley (1981) Optimal strategies for repeated games. *Advanced Applied Probability* 13: 415-428.

Griffin, P. (1985) Different measures of win rates for optimal proportional betting. *Management Science* 30: 1540-1547.

Hakansson, N.H. and B.L. Miller (1975) Compound-return mean-variance efficient portfolios never risk ruin. *Management Science* 22: 391-400.

MacLean, L. C, R. Sanegre, Y. Zhao and W. T. Ziemba (2002) Capital growth with security. *Journal of Economic Dynamics and Control* (in press).

Mossin, J. (1968) Optimal multiperiod portfolio policies. *Journal of Business* 41: 215-229.

Samuelson, P.A. (1971) The "fallacy" of maximizing the geometric mean in long sequences of investing or gambling. *Proceedings National Academy of Science* 68: 2493-2496.

Thorp, E.O. (1975). Portfolio choice and the Kelly criterion. *Stochastic Optimization Models in Finance* In. W.T. Ziemba and R.G. Vickson. Academic Press, New York.

Ziemba, W.T. and D.B. Hausch (1986) *Betting at the Racetrack*. Dr. Z. Investments, Los Angeles.

[1] For those who wish to chase down all these papers, see my cited papers with MacLean.

W

Loading preview . . .

28th Annual

2011 World Cup

CHAMPIONSHIP OF FUTURES AND FOREX TRADING®

WorldCup Championships.com

OFFICIAL ENTRY AGREEMENT

The undersigned wishes to compete in the 2011 World Cup Championship of Futures and Forex trading sponsored by WorldCupAdvisor.com ("Sponsor") and has completed an Account Application to open a futures, forex or combined futures/forex trading account with a broker authorized by WorldCupAdvisor.com. An authorized broker ("Authorized Broker") is any broker who introduces a World Cup Championship account to PFGBEST.com ("Carrying Broker") on a fully disclosed basis. Carrying broker may, in its sole discretion, accept the account. If accepted, it is agreed that the undersigned and his Account Manager, if any (collectively referred to herein as "Entrant") shall be entitled to participate in the 2011 World Cup Championship of Futures & Forex Trading ("The Championship") subject to the following conditions:

1. CHAMPIONSHIP TRADING ACCOUNT

Entrant agrees to deposit a minimum of $15,000 US, per entry, in a futures, forex or combined futures/forex trading account satisfactory to Carrying Broker. Additional funds may be deposited at any time, including to meet a margin call as specified in paragraph 6, and will be added to the initial deposit to calculate Total Funds Deposited. Entrant understands and agrees that no funds may be withdrawn from the account except upon Entrant's termination of participation. Accordingly, Entrant should consider depositing more than $15,000 if Entrant's selected markets or trading style could require additional margin. Entrant may open more than one account at the beginning of the Championship or at any time during the Championship Period. Each account must be separately funded. The same Account Application may be used to establish multiple accounts.

2. CHAMPIONSHIP PERIOD

Entrant agrees to begin trading on or after January 3, 2011. The Championship trading period will end as of the close of business on December 30, 2011, subject to paragraph 12. Open positions need not be liquidated for purposes of determining winners of the Championship.

3. PARTICIPATION AND CHAMPIONSHIP AWARDS

Each of the top five profitable finishers as determined in paragraph 8 will be authorized to trade an Award Account in the amount of $50,000. Carrying Broker shall establish a segregated trading account of Nominal Value $50,000 on its books for each of the top five profitable finishers and shall issue a limited power of attorney over the Award Account to each Entrant. The Award Account winners shall manage the Award Account in the same manner employed during the World Cup Championship. Each Award Account winner shall earn a quarterly incentive fee on net profits in the Award Account above a high water mark. Award Account winners agree to execute any and all documents necessary or advisable regarding the establishment and ongoing management of their Award Account, an affidavit of compliance with this Agreement and any liability or publicity releases that may from time to time be requested by Carrying Broker. Carrying Broker may close any and/or all Award Account(s) at any time and for any reason in its sole discretion. However, Carrying Broker intends to maintain the Award Account(s) for as long as the Award Account generates returns that are commensurate with the performance volatility and risk exhibited by the Award Account. The top three profitable finishers will receive Championship Awards as set forth in "Winners' Prizes" published on the www.worldcupchampionships.com web site. The top profitable Entrant shall also receive a personalized copy of the coveted Bull & Bear Trophy, and the second and third-place finishers shall receive a personalized crystal Bull & Bear trophy. In consideration of participation in the Championship, Entrant irrevocably authorizes Sponsor, in its sole discretion, to obtain and print, publish, televise or otherwise utilize his, her or their names, photographs, account statements, and descriptions of World Cup participation in connection with this or future Championships and with other promotions deemed appropriate by the sponsor. Sponsor may compensate Entrant at its discretion.

4. REPRESENTATIONS OF ENTRANT AND ACCOUNT MANAGER

The Entrant represents that he and his Account Manager (if applicable) are of legal age in the states in which they reside and that neither of them nor any member of their household nor any of their partners or shareholders is related to principals or employees of the Sponsor or Carrying Broker. Each represents that the Account Manager is properly licensed (if required by law or regulation) or otherwise exempt from registration. Entrant's account shall be non-discretionary unless the Account Manager also signs and is a party to this agreement.

5. CHAMPIONSHIP TRADING REGULATIONS

All trading shall be conducted in accordance with a separately executed Customer Agreement. Entrant agrees to the following Championship rules and regulations, which may limit those set forth in the Customer Agreement. The purpose of the Championship rules is to ensure that every Entrant will have the same opportunity for success. Sponsor reserves the right to amend, waive, or interpret any rule if, in its sole discretion, to do so would be in the best interests of the Championship.

a. Entrant has the option at any time to withdraw from the Championship and either continue to trade pursuant to the terms of the Customer Agreement or cease trading.

b. Entrant agrees to place all orders through Carrying Broker's online order entry platform or telephone order desk. Entrant agrees to call the desk only when placing or changing orders.

c. The Carrying Broker is not obligated to give Entrant any advice or market information except the last price traded and the margin requirements for existing or contemplated positions. Carrying Broker, in its sole discretion, may terminate the participation of any Entrant for Entrant's failure to enter orders in a timely, consistent and professional manner.

d. Entrant agrees to liquidate all open positions maturing in a current futures month at least one day prior to first notice day for long positions and five days prior to the last trading for short positions. Carrying Broker may in its discretion effect such liquidation if Entrant has not given liquidating orders by the second day prior to the first notice day in the case of a long position or by the sixth day before the last trading day in the case of a short position. Options must be exercised in accordance with the procedures set forth in the Customer Agreement.

e. Accounts will be charged all-in brokerage commissions of $12 per round-turn contract in electronic futures markets, and $12 per round-turn contract plus applicable fees in open outcry futures markets. Accounts will be charged $60 per million on forex trades. Orders placed by phone will be charged an additional $10.

f. Trading in the Championship will be limited to listed futures contracts and options thereon on any exchange available through Carrying Broker, and forex transactions effected through Carrying Broker.

g. In the event of errors in order entry or execution, the determination of Sponsor and Carrying Broker shall be final with regard to Championship standings.

h. Trading must be conducted in Entrant's Championship account(s). A minimum of ten (10) round-turn trades of any contract number for futures trades or any base currency size for forex trades, per account, must be placed during the Championship Period in order for Entrant to qualify for a Championship Award for that account.

i. Each Entrant agrees that Sponsor has the right to remove or bar any Entrant from the Championship, who, in its sole judgment, would tend to dishonor the Championship, has violated any rule, law, or regulation pertaining to futures or forex trading or who has attempted to benefit from any collusive or other trading irregularity. Sponsor shall have the same right to remove or bar Account Managers.

j. Entrant authorizes Carrying Broker to provide Sponsor with the ability to view activity in Entrant's account for the purpose of monitoring Championship performance.

k. Entry in the Championship is void where prohibited by law. Winners will be responsible for any taxation on awards.

6. MARGIN CALLS AND LIQUIDATION OF POSITIONS

Initial margin for new positions and maintenance margin for existing positions must be maintained in accordance with Carrying Broker's requirements, which may be adjusted from time to time without prior notice. Entrants may deposit additional funds for any reason, including to meet a margin call. Additional funds deposited will be added to the initial deposit to calculate Total Funds Deposited. Entrants, with Carrying Broker's consent, may liquidate positions in order to meet a margin call.

7. PARTICIPATION TERMINATION

If the total equity in an Entrant's account at the close of any trading day falls below $2,500, Broker may terminate that account's participation in the Championship. Carrying Broker may also terminate Entrant's participation as a result of procedures in the separately executed Auto Liquidation agreement. A new account may be established with new funds at any time.

8. TOP TRADERS DETERMINATION

The top traders for prizes, trophies and Award Accounts as specified in paragraph 3 will be determined on the basis of net return. All futures, options and forex positions will be marked to the market on the close of business on December 30, 2011 in order to determine the account's "Ending Equity." For the purposes of the Championship results, Ending Equity will be ledger balance plus or minus open equity. The Entrants with Ending Equity that shows the highest percentage increase over their Total Funds Deposited (initial deposit plus additional deposits, if any) will be the winners. Percentage increase for all prizes will be calculated by taking the Ending Equity on Dec. 30 and dividing it by Total Funds Deposited. Only profitable accounts will qualify for prizes, trophies and Award Accounts.

9. USE OF CHAMPIONSHIP FOR PROMOTIONAL PURPOSES.

WorldCupAdvisor and World Cup Championship of Futures & Forex Trading are trademarks and registered trademarks. Entrant shall not and agrees not to, either alone or in concert with others, use the trademarks (or derivations thereof) or his participation in the Championship for promotional purposes unless approved in advance in writing by Sponsor. If Entrant violates this provision, Entrant may be disqualified from competing in the Championship or future Championships at Sponsor's sole discretion. If found by Sponsor to be in violation, Entrant agrees to pay all legal fees and damages incurred by Sponsor to enforce this provision, and, if applicable, return to Sponsor any award(s) received as liquidated damages.

10. COMMUNICATION DELAYS

Broker shall not be responsible for any delays in the acceptance or transmission of orders due to a breakdown or failure of transmission, computer (hardware, software or interfaces) or communication facilities, or for any other cause beyond their reasonable control or anticipation.

11. INDEMNIFICATION

Entrant agrees to indemnify Sponsor, Carrying Broker and Authorized Broker and hold them harmless from and against any and all liabilities, losses, damages, costs and expenses, including attorney's fees, incurred by any of them resulting from: misrepresentations, breach of any provision of this agreement, the trading in Entrant's account(s), or legal action brought by Entrant and adjudicated in favor of Sponsor or Broker.

12. POSTPONEMENT, CANCELLATION AND ACCEPTANCE

Sponsor reserves the right to modify the length of the trading period, to postpone the starting date or to cancel the Championship if, in its sole discretion, it determines that such action is reasonable or necessary. This agreement shall not be deemed accepted until approved by Sponsor. Sponsor and Carrying Broker, in their sole discretion, may reject any Entrant's application for any reason and return it together with all funds submitted.

______________________	______________________
Account Holder's Name	Account Holder's Signature
______________________	______________________
Joint Account Holder's Name	Joint Account Holder's Signature
______________________	______________________
Street Address	Signature of Account Manager, if any
______________________	__________@__________
City, State, Zip	E-mail address
______________________	______________________
Telephone Number	Initial Deposit

Betting with the Kelly Criterion

Jane Hung

June 2, 2010

Contents

1 Introduction

Gambling in all forms, whether it be in blackjack, sports, or the stock market, must begin with a bet. In this paper, we summarize Kelly's criterion for determining the fraction of capital to wager in a gamble. We also test Kelly's criterion by running simulations.

In his original paper, Kelly proposed a different criterion for gamblers. The classic gambler thought to maximize expected value of wealth, which meant she would need to invest 100% of her capital for every bet. Rather than maximizing expected value of capital, Kelly maximized the expected value of the utility function. Utility functions are used by economists to value money and are increasing as a function of wealth under the assumption that more money can never be worse than less [1]. Kelly took the base 2 logarithm of capital as his utility function [2], but we will use the base e logarithm (the natural log) instead.

2 Kelly Criterion

The following derivation is modified from Thorp [1]. We assume that the probability of events are known and independent and that the probability of a win is p $(1 > p > 1/2)$ and the probability of a loss is $q = 1 - p$. Suppose a fraction f $(0 < f < 1)$ of the capital is bet each turn and W_n and L_n represent the number of wins and losses after n bets, respectfully. Rather than even payoff (i.e., a win of 1 unit per unit bet per win), we consider the more general scenario that b units are won per unit bet per win and a units are lost per unit bet per loss. Given initial capital X_0, the capital after n bets is

$$X_n = X_0(1 - af)^{L_n}(1 + bf)^{W_n}.$$

Now define

$$g(f) = log\left(\frac{X_n}{X_0}\right)^{\frac{1}{n}} = \frac{1}{n}\left(L_n \cdot log(1 - af) + W_n \cdot log(1 + bf)\right),$$

the exponential rate of increase per trial. The expected value of $g(f)$ is

$$G(f) = E(g(f)) = q \cdot log(1 - af) + p \cdot log(1 + bf)$$

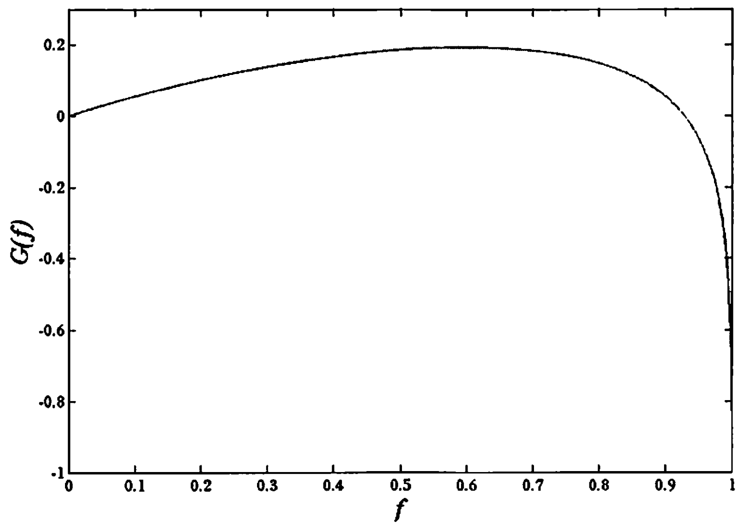
because the ratio of expected wins or losses to trials is given by the probabilities p and q, respectively. We want to maximize $G(f)$ because

$$G(f) = E(g(f)) = E\left(\frac{1}{n} \cdot log(X_n) - \frac{1}{n} \cdot log(X_0)\right),$$

so maximizing $G(f)$ would in turn maximize $E(log(X_n))$, the expected value of the logarithm of wealth. A critical point of $G(f)$ can be found by setting the derivative to 0:

$$G'(f) = -\frac{aq}{1 - af} + \frac{bp}{1 + bf} = 0$$

Figure 1: Expected Value of Logarithm of Wealth vs. Bet as a Fraction of Wealth



$$= \frac{bp - aq - abf}{(1+bf)(1-af)} = 0,$$

so the critical point is at

$$f = f^* = \frac{bp - aq}{ab}.$$

Notice that $f^* \neq \frac{1}{a}$, so $G'(f)$ is defined at f^*, and the critical point is a zero of $G'(f)$ there. Since

$$G''(f) = -\frac{a^2 q}{(1-af)^2} - \frac{b^2 p}{(1+bf)^2} < 0,$$

f^* is a local maximum. And because $G(0) = 0$ and $\lim_{f \to 1^-} G(f) = -\infty$, the maximum of $G(f)$ is at f^*. Figure 1 shows the plot of $G(f)$ as a function of f. For this function, we set $p = 0.8$, $q = 0.2$, and $a = b = 1$. The maximum occurs at $f^* = p - q = 0.6$.

3 The Stock Market

Kelly criterion can be applied to the stock market. In the stock market, money is invested in securities that have high expected return [3]. The following derivation is modified from Thorp [1]. Since there is not a finite number of outcomes of a bet on a security, we must use continuous probability distributions. Let X

be a random variable that denotes the return per unit, and suppose

$$P(X = \mu + \sigma) = P(X = \mu - \sigma) = \frac{1}{2}.$$

Then the expected value E(X)=μ, and the variance of X is σ^2 (with standard deviation σ). Suppose the initial capital is Y_0 and the bet as a fraction of wealth is f. Then the capital $Y(f)$ is given by

$$Y(f) = Y_0(1 + (1 - f)r + fX),$$

where r is the rate of return of capital invested elsewhere. Using the probability assumptions, this means

$$G(f) = E\left(log\left(\frac{Y(f)}{Y_0}\right)\right)$$

$$= \frac{1}{2}log\,(1 + (1 - f)r + f(\mu + \sigma)) + \frac{1}{2}log\,(1 + (1 - f)r + f(\mu - \sigma)).$$

If there are n time steps of equal length in the time interval, then we have X at each of those steps, X_i, with i=1, 2,..., n. Also,

$$P\left(X_i = \frac{\mu}{n} + \sigma n^{-\frac{1}{2}}\right) = P\left(X_i = \frac{\mu}{n} - \sigma n^{-\frac{1}{2}}\right) = \frac{1}{2}$$

given that we want the same total μ, σ^2 and r. Then we have

$$\frac{Y_n(f)}{Y_0} = \prod_{i=1}^{n}\left(1 + (1 - f)\frac{r}{n} + fX_i\right)$$

and

$$G_n(f) = E\left(log\left(\frac{Y_n(f)}{Y_0}\right)\right) = E\left(\sum_{i=1}^{n} log\left(1 + (1 - f)\frac{r}{n} + fX_i\right)\right)$$

$$= \sum_{i=1}^{n}\frac{1}{2}log\left(1 + (1 - f)\frac{r}{n} + f\left[\frac{\mu}{n} + \sigma n^{-\frac{1}{2}}\right]\right) + \frac{1}{2}log\left(1 + (1 - f)\frac{r}{n} + f\left[\frac{\mu}{n} - \sigma n^{-\frac{1}{2}}\right]\right)$$

$$= \frac{n}{2}log\left(\left(1 + (1 - f)\frac{r}{n} + f\frac{\mu}{n}\right)^2 - f^2\sigma^2 n^{-1}\right).$$

Now we expand $G_n(f)$ as a Taylor series around $f = 0$. Calculating the derivatives of $G_n(f)$, we get

$$G_n(0) = n \cdot \frac{r}{n} + O(n^{-\frac{1}{2}})$$

$$\frac{dG_n(0)}{df} = n \cdot \frac{\mu - r}{n} + O(n^{-\frac{1}{2}})$$

$$\frac{d^2G_n(0)}{df^2} = n \cdot -\frac{\sigma^2}{2n} + O(n^{-\frac{1}{2}})$$

and

$$\frac{d^k G_n(0)}{df^k} = O(n^{-\frac{1}{2}})$$

for k $\geq$ 3. Then G_n can be expressed as

$$G_n(f) = r + (\mu - r)f - \sigma^2 \frac{f^2}{2} + O(n^{-\frac{1}{2}}).$$

To make this continuous, we allow $n \to \infty$; thus G_n becomes

$$G_\infty(f) = r + (\mu - r)f - \sigma^2 \frac{f^2}{2}.$$

Notice that $f < 0$ is allowed and is equivalent to taking a short position. This G_∞ is an instantaneous growth rate, so adjustments must be made when Y_n undergoes a change. Using the method in section 2, we find that the optimal betting fraction, f^*, is

$$f^* = \frac{\mu - r}{\sigma^2}.$$

4 Simulations

Using MATLAB, we simulated betting with two different strategies: one using the Kelly Criterion and another with constant betting. The scenario is smplified such that the probability of a win and a loss are known and constant. This may be realistic in the case of a very consistent sports team for example. The parameters given are

probability of winning the bet $p = 0.55$,

probability of losing the bet $1 - p = q = 0.45$,

units won per unit bet per win $b = 10/11$,

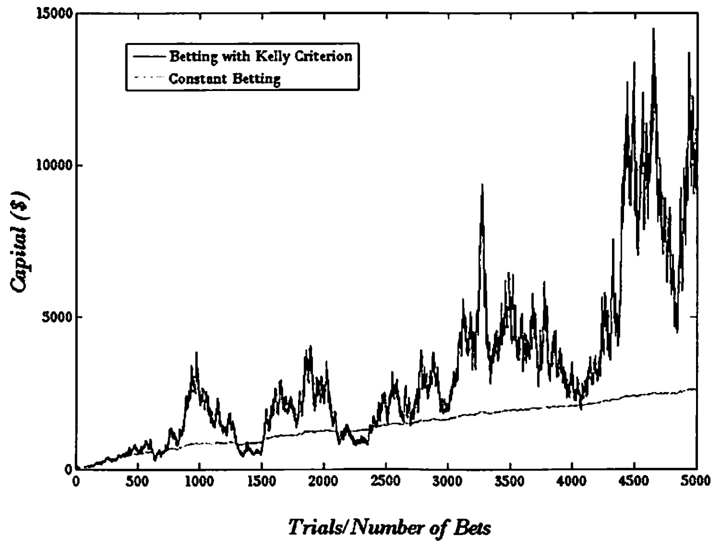
units lost per unit bet per loss $a = 1$,

number of trials $n = 5000$,

and initial capital $X_0 = \$100$.

The rand command in MATLAB was used to generate random numbers for determining the outcome of each trial; this command returns pseudorandom numbers from a uniform distribution. The results are shown in Figure 2.

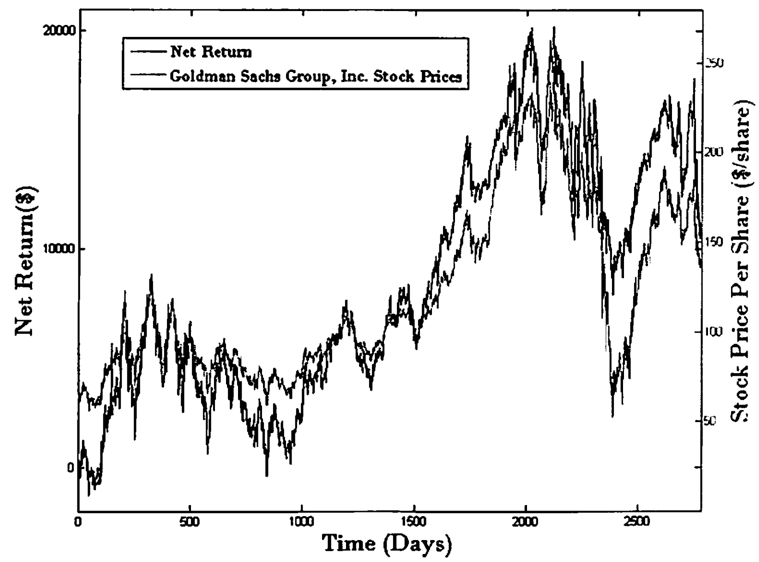
Figure 2: Capital Through 5000 Bets: Betting with the Kelly Criterion vs. with constant bets.



From the graph, betting with the Kelly Criterion clearly has an advantage over constant betting. After 5000 bets, betting with the Kelly Criterion yields a total capital of between $5000 and $10000 (a percent increase of capital of over 4900%) while constant betting yields a total capital of around $2500 (a percent increase of capital of about 2400%). However, unlike the Kelly Criterion curve, constant betting showed a roughly linear trend line; the fluctuations cannot be measured readily by glance. With the Kelly Criterion, the fluctuation is orders of magnitude different though the overall upward trend is above that of constant betting. Noticeable drops and gains of thousands of dollars within 100 bets are evident from looking at the Kelly Criterion graph. In addition, betting with the Kelly Criterion may occasionally be worse than constant betting even after several thousand bets.

The number of bets considered here should also be discussed. Betting 5000 times may be unrealistic for most. If 3 bets were made every week, it would take around 32 years to reach 5000. During this time, even a consistent team would likely not carry the same win percentage! For the short term, it may be better to look at the performance of betting with the Kelly Criterion through 150 bets (1 year's worth of betting). In this interval, the Kelly Criterion seems virtually identical to constant betting. There does not seem to be a significant increase in capital during that time with either method. Appreciable differences are seen only at around 1000 bets, so in order to experience the advantage of using the Kelly Criterion, a bettor should start with more capital, make more bets, or be

Figure 3: Net Return Through 10 Years: Investing in Goldman Sachs using the Kelly Criterion.



willing to wait a long time. From this simulation, we see that betting with the Kelly Criterion is effective after many trials but also quite volatile.

Use of the Kelly Criterion is further investigated through application to the stock market. The closing stock prices of Goldman Sachs Group, Inc. (GS) from May 30, 1999 to May 24, 2010 were obtained [4] and used as the data. In this period, the stock rose from 64.19 to 136.69. Since stocks typically experience many highs and lows, one single mean and standard deviation value cannot represent the behavior of the stock through 11 years accurately. Thus, the data was split into nineteen 146 day blocks, and the mean and standard deviation of each block was found. The optimal fraction for each block could then be calculated. The parameters given are

$$\text{return rate of other investments } r = 0.00,$$

$$\text{number of days} = 2774,$$

$$\text{initial investment } Y_0 = \$10000.$$

The stock price is the price per share, so the number of shares for day k was given by the investment for that day as suggested by the Kelly Criterion divided by that day's stock price. Fractional shares were allowed. The subsequent value of those shares was the product of the number of shares for day k and the stock price on day $k + 1$. To simplify matters, the rate of return of the uninvested wealth was set to zero. Hence the total wealth was the sum of the uninvested wealth and the value of the invested wealth. This yielded the net return when subtracted by the initial investment. It should be noted that the fraction of wealth to invest was limited to ≤ 1 so that we did not have to deal with short selling or debt. The results can be seen in Figure 4.

The results are similar to those found in the case of sport betting. The net return after 11 years is about \$10000, which is 100% of the initial investment. While investing higher fractions of wealth would increase the net return slightly, that is an extremely risky strategy when the future stock price is unknown. The Kelly Criterion clearly involves nontrivial risk, as evidenced by the negative return within the first 100 days; however, the risk is reduced by the changing of fraction of wealth invested.

This was a simplified example, so the actual outcome would differ if, e.g., the uninvested wealth were put into a risk free security, or if short selling or debt were considered so that the fraction of wealth invested could be above 1. Still, this simulation provides insight into how the Kelly Criterion might perform when used on the stock market.

5 Conclusion

The Kelly Criterion can be utilized to find the optimal bet size for a wager. Not only can Kelly Criterion be used for sports betting and casino games, it can also be used in the stock market. We derived the optimal bet size expression for a situation with only two outcomes and discrete time steps. Furthermore, we used continuous probability distributions to find the optimal bet size expression in a situation where securities may be bought or sold. Finally, we simulated a betting situation using MATLAB and compared the results of betting with the Kelly Criterion to constant betting. This was expanded to investing in the stock market. We found that the Kelly Criterion is effective, but initial capital should be high and/or a great deal of time should be allowed for the final capital to reach substantial amounts. In this way, the Kelly Criterion is impractical and so is not applied in many situations.

References

[1] Thorp, E. The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market. Paper presented at: The 10th International Conference on Gambling and Risk Taking. 1997.

[2] Kelly, J.L. A New Interpretation of Information Rule. Bell System Technical Journal. 35. (1956): 917-926

[3] Thorp, E. O. Optimal Gambling Systems for Favorable Games. Revue De L'Institut International De Statistique. 37. 3 (1969): 273-293.

[4] Goldman Sachs Group, Inc. Historical Prices. Google Finance. http://www.google.com/finance/historical?cid=663137&startdate=May+30%2C+1999&enddate=May+29%2C+2010. May 29, 2010.